MATH'S CONVERSATION IN ENGLISH

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Volume 1:

Integer Discussion
Fraction Discussion
Decimals and Percents Discussion

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1. Integer Discussion

Pupil : What exactly is an integer?

Teacher : Well, you know what the natural numbers are, right?

Pupil : Yes, the natural numbers are the counting numbers, 1, 2, 3, 4, 5, and so on.

Teacher : That's right. The integers are a set of numbers including the natural numbers as well as zero and the negative numbers.

Pupil : The negative numbers?

Teacher : Yes, the negative numbers are the opposite of the natural numbers, also called positive numbers, and are usually expressed as -x (if x is a natural number). Think about it like this: if the number 5 is greater than zero by five units, then – 5 is less than zero by five units.

Pupil : So that means that a positive number is always greater than its negative.

Teacher : Exactly, and furthermore it means that any positive number is greater than any negative number. It is also interesting to note that as a positive number gets larger its negative counterpart gets smaller.

Pupil : Then that means – 8 is less than – 3 since 8 is greater than 3.

Teacher : It seems like you understand. Now, can you think of a situation where you would use negative numbers?

Pupil : Wouldn't you use negative numbers if, for instance, you owed more money than you had in your account. If you were in debt Rp 300,000.00 then the amount of money you have could be represented as – Rp 300,000.00.

Teacher : That's good, but can you think up one more example just to be certain you understand.

Pupil : Yes, in golf negative numbers are used to describe how many strokes a player is below par. For instance, if you were 5 strokes below par, your score could be expressed as – 5, or "five under par".

1.1 Integer Addition and Subtraction Discussion

Teacher : Now that we have established a definition for the Integers, we can perform operations on them.

Pupil : What kind of "operations"?

Teacher : Well, the easiest one would be addition. Consider this example: a friend gives you 3 pieces of candy. If you already had 2 pieces of candy, then how many pieces would you have?

Pupil : If I had three pieces and was given two more, then I would have 5 pieces of candy.

Teacher : Yes, that is right. You just added two integers. Now, to express this operation we would write, 3 + 2 = 5, which represents 3 pieces of candy plus (or in addition to) 2 pieces of candy equals 5 pieces of candy. Now what if you had three pieces of candy and someone gave you two more?

Pupil : Hmm, if I had 3 pieces and someone gave me 2 more, then I would end up with 5 pieces of candy, again. So if 2 + 3 = 5 and 3 + 2 = 5, then does 2 + 3 = 3 + 2?

Teacher : Yes, in fact, for any two integers b and c, $b + c = c + b$. You can add them in either order, we call this the Commutative Law of Addition. Also, if you are adding more than two numbers, you may add them in any order you like, i.e. if you were adding $b + c + d$, then you could first add b to c, $(b + c) + d$, or you could first add c to d, $b + (c + d)$. We call this the Associative Law of Addition (parentheses are used to emphasize the order of the operations performed).

Pupil : The Integers also include the Negative Numbers, right? What happens when you add a positive number to a negative number, or two negative numbers together?

Teacher : Well, let's say we had two positive numbers b and c. Then $b + (-c)$ would be the same as writing $b - c$, or b minus c. We call this subtraction. Let's return to our candy example for a moment. What if you started off with 5 pieces of candy and then someone took away 2 pieces?

Pupil : If I had 5 pieces and someone took away 2 pieces I would have 3 pieces left. So 5 - 2 = 3, right?

Teacher : Right, however, it is important to know that 5 - 2 is not the same as 2 - 5. To subtract a larger number from a smaller number simply start with the smaller number and count down. Once you hit zero, the next number will be -1, then -2, and so on.

Pupil : So to figure out what 2 - 5 equals I would count down five numbers from 2: 1, 0, -1, -2, -3. So 2 - 5 = -3, which is definitely not equal to 5 - 2 = 3.

Teacher : You certainly seem to understand addition and subtraction. Notice that if you subtracted b units from b you would end up back at zero. For any number b, $b - b = 0$. Now I have one last question. What do you think would happen if you added or subtracted 0 from a number?

Pupil : Well if you did not add anything to a number, it wouldn't change, so $b + 0 = b$ if b is any integer. The same would be true if you did not subtract anything from a number, so $b - 0 = b$.

Teacher : Yes, and since adding or subtracting 0 from any number changes nothing, we can drop it out of the computation completely. This leaves us with $b = b$, for any b, which we call the identity equation. It tells us that any number b is always equal to itself.

1.2 Integer Multiplication Discussion

Pupil : So now that I understand addition and subtraction, are there more operations?

Teacher : Yes there are. The next operation is called multiplication. We write multiplication problems in the form a x b, or a times b. For an example we will consider 5 x 3. What this means is that 5 is being added to itself 3 times (5 + 5 + 5), but a better way to think about is that it means 5 groups of 3 units each. Therefore, to solve 5 x 3 you would count the number of units in each group.

Pupil : I feel a little confused.

Teacher : Alright, we will use an example you can visualize. Picture 5 separate plates, each with 3 quarters on them. How many quarters are there all together?

Pupil : Well, if there are 3 quarters for each of the 5 plates, then there are 15 quarters all together. So 5 x 3 = 15.

Teacher : Exactly. Now, what if we had 3 plates, each with 5 quarters?

Pupil : Then there would be 5 quarters for each of the 3 plates, meaning 15 quarters altogether. So 3 x 5 = 15. Does that mean that multiplication is commutative like addition?

Teacher : Yes, multiplication is commutative. For any a and b, $a \times b = b \times a$. It is also associative. If you are multiplying three numbers, then $(a \times b) \times c = a \times (b \times c)$. Now, what do you think would happen in the case where a number was multiplied by 0 or 1? If you have trouble just try to visualize what it means.

Pupil : Hmm, if you had any number of plates p, each with 0 quarters, you would have 0 quarters altogether. So any number times 0 equals 0. Now, if you had 1 plate with q quarters, then there would be q quarters altogether. So any number times 1 is just the number itself.

Teacher : That is absolutely right. You thought through that very well.

Pupil : This isn't so hard, but what about negative numbers? I know how to add them, but how do you multiply negative numbers?

Teacher : Well, first I should explain a little more about what a negative number is. Even though the negative numbers are less than zero, this does not mean that they are less than nothing. A negative number is merely a negation of a positive number. For instance, what would happen if you traveled one mile east and then backtracked one mile west?

Pupil : You would end up where you started.

Teacher : Exactly, the one mile you traveled west negated the mile you traveled east. Another way to express your movement west would be to say you traveled -(one mile east), or negative one mile east. So we can think of the negative sign as an operator, just as the multiplication sign is an operator. Its operation would be to replace positive units with negative units.

Pupil : Where exactly is this all leading us?

Teacher : Precisely where we want to be. Remember that a positive unit plus a negative unit always equals zero, therefore, the relation between positive and negative is reciprocal. This means that the negative of a positive is a negative and that the negative of a negative is a positive. Knowing this, we can now move on to multiplication by negative numbers.

Based off the definition I gave you earlier, what do you think 2 x (-3) equals?

Pupil : Well, 2 x (-3) would be 2 groups, each with (-3) units, so that would be (-6) units.

Teacher : Good, that one was tricky, but mostly straightforward. How about this one: What is (-2) x 3?

Pupil : That would be (-2) groups, each with 3 units. How do you count negative groups, though?

Teacher : First, let me rephrase it as negative 2 groups, each with 3 units. Now, a negative group is merely a group that would negate a positive group. What kind of group would negate a group of 3 units?

Pupil : A group of 3 units would be negated by a group of (-3) units, right?

Teacher : Yes, therefore (-2) x 3 = 2 x (-3), which we already know is equal to (-6) units. Now what do you think (-2) x (-3) equals?

Pupil : Well, that would be negative 2 groups, each with 3 negative units. So I need to know what kind of group would negate a group of 3 negative units . . . a group of 3 positive units! So (-2) x (-3) = 2 x 3 = 6. I think that's right.

Teacher : That's exactly right. So what you can see from these few examples is that a negative number times a positive number will give you a negative number and a negative number times a negative number will give you a positive number.

1.3 Integer Division Discussion

Teacher : Now we should look at the division operation. A good way to visualize what will be going on is to think of a piece of rope. If we have a piece of rope that is 12 units long, how many 4 unit pieces can we cut from it?

Pupil : If we cut 4 unit pieces of rope from a 12 unit long rope, we will end up with three pieces.

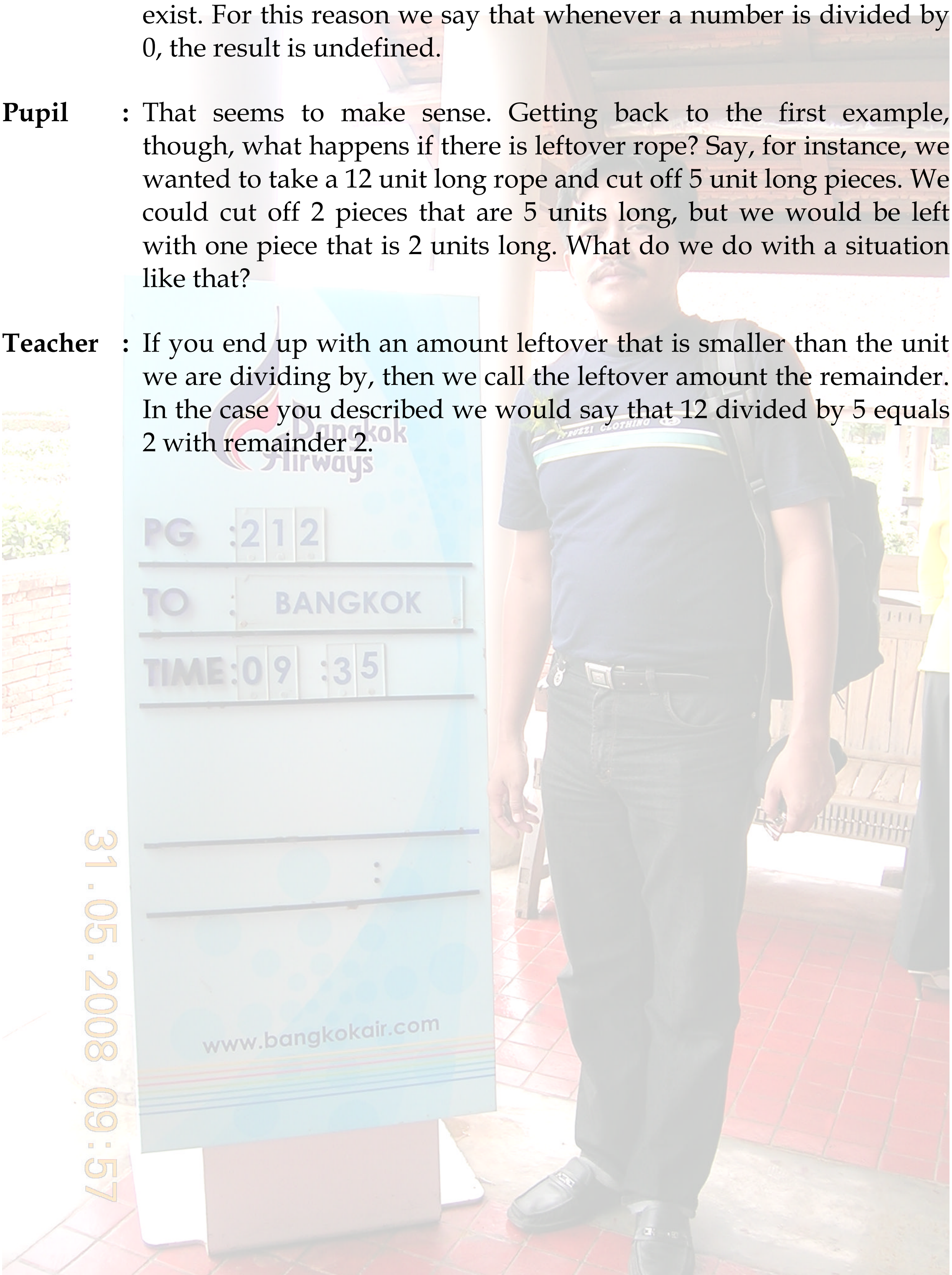
Teacher : Correct. To express this mathematically we would write $12 \div 4$, or 12 divided by 4. What do you think happens when we divide a number by 1 or 0? Be careful, though, 0 is tricky.

Pupil : If a piece of rope were r units long we cut off 1 unit pieces at a time, then we could get r pieces. So any number divided by 1 is that number. If you divided rope of length *r* into pieces that were each 0 units long. . . wait, can a piece of rope without length exist?

Teacher : No, that would be like having an object without mass, it cannot exist. For this reason we say that whenever a number is divided by 0, the result is undefined.

Pupil : That seems to make sense. Getting back to the first example, though, what happens if there is leftover rope? Say, for instance, we wanted to take a 12 unit long rope and cut off 5 unit long pieces. We could cut off 2 pieces that are 5 units long, but we would be left with one piece that is 2 units long. What do we do with a situation like that?

Teacher : If you end up with an amount leftover that is smaller than the unit we are dividing by, then we call the leftover amount the remainder. In the case you described we would say that 12 divided by 5 equals 2 with remainder 2.



2. Fractions Discussion

Pupil 1 : I thought we were studying fractions now, so what do fractions have to do with a carton of eggs?

Teacher : Suppose we were mixing up a batch of blueberry muffins and the recipe called for four eggs. How would we represent four of the eggs in the carton as a fraction?

Pupil 2 : Well, twelve eggs come in a carton, so the fraction would be $\frac{4}{12}$, or four parts out of twelve.

Teacher : Good. And if we look in the carton after we take out the eggs, we see an example of what four out of twelve looks like.

Teacher : And if we remove two more eggs, we see what six out of twelve, or $\frac{6}{12}$ eggs looks like.

$$\frac{6}{12} = \frac{1}{2} = \frac{\text{Nominator}}{\text{Denominator}} = \frac{\text{Number of parts}}{\text{Entire number}}$$

Pupil 1 : Yes, six out of twelve is exactly half of the eggs.

Teacher : That's right. We can reduce the fraction $\frac{6}{12}$ to a half, or $\frac{1}{2}$. These two fractions are the same.

Pupil 1 : Can we reduce the fraction $\frac{4}{12}$?

Teacher : In the same way that six is half of twelve, or $\frac{1}{2}$, four is one third of twelve, or $\frac{1}{3}$. Both numbers can be divided by four: 4 divided by 4 is 1; 12 divided by 4 is 3, so the fraction is $\frac{1}{3}$.

$$\frac{4}{12} = \frac{4 / 4 = 1}{12 / 4 = 3} \quad \text{so} \quad \frac{4}{12} = \frac{1}{3}$$

Pupil 2 : What fraction is it if we remove all the eggs?

Pupil 1 : There would be no eggs left, $\frac{0}{12}$, or zero, so there wouldn't be a fraction, because there can not be a part of nothing, right?

Teacher : Good thinking. Nothing is always nothing, so we can't remove some of it, or divide it into parts.

Pupil 2 : Then what if we didn't take out any parts at all; what if we left all the eggs in the carton?

Pupil 1 : The fraction would be $\frac{12}{12}$, which would be the same as ...

Teacher : One. Twelve out of twelve eggs equals one entire carton of eggs. Any fraction that has the same numerator and denominator equals the number 1.

$$\frac{12}{12} = 1$$

Pupil 2 : OK, I understand that now. I'm ready to make blueberry muffins!

2.1 Comparing Fractions Discussion

Teacher : Any number can be made into a fraction, and many fractions can be reduced.

Pupil : How can a whole number, such as 4 or 5 or 20, be made into a fraction?

Teacher : By putting the number 1 as the denominator of any number, that number can be expressed as a fraction, but still keep its value.

$\frac{5}{1} = 5$, and $\frac{20}{1} = 20$

Pupil : Could I also write a fraction that equals 5 if I expressed it as $\frac{20}{4}$? Because $\frac{20}{4} = 5$.

Teacher : Yes, that also works, but notice that the fraction $\frac{20}{4}$ can be reduced by dividing the numerator and denominator by 4. So the simplest way to express the value 5 as a fraction is by writing $\frac{5}{1}$.

Teacher : This is a helpful thing to remember when working with fractions and whole numbers, such as adding, subtracting, multiplying or dividing them. Sometimes it helps to write the whole numbers as fractions when solving the problem.

Pupil : The fractions $\frac{20}{4}$ and $\frac{5}{1}$ are equal to each other?

Teacher : Yes. Which fraction would be larger, $\frac{3}{4}$ or $\frac{4}{5}$?

Pupil : I am not sure. Can I try drawing a picture to help me see the solution?

Teacher : Good idea. First we could start by drawing two bars of equal length, but breaking one bar into four equal sections, and the other into five equal sections.

Pupil : I see, then we colour in as many pieces as are in the numerator: 3 for the bar with four segments and 4 for the one with five. They now show what $\frac{3}{4}$ and $\frac{4}{5}$ looks like.

Pupil : OK, now I can easily see that the fraction $\frac{4}{5}$ is larger than $\frac{3}{4}$.

Teacher : Good. So which fraction is smaller, $\frac{1}{6}$ or $\frac{1}{8}$?

Pupil : Well, let me start by drawing two bars of equal length, dividing them into as many pieces as are in the denominator, and .....

Can you finish this Pupil's work? Which fraction is smaller?

2.2 Fraction Adding and Subtracting

Pupil : When I add fractions that have the same denominator, such as $\frac{1}{5}$ + $\frac{2}{5}$, I know that the answer is $\frac{3}{5}$. But what do I do when the denominators are different?

Teacher : The first thing is to find the lowest common denominator, the smallest number that both denominators divide into evenly. If I am adding $\frac{3}{4} + \frac{1}{6}$, what is the lowest common denominator?

Pupil : Well, it looks to me like it would be 12. Both 4 and 6 can divide into 12.

Teacher : Exactly! So you rewrite the problem to look like this:

Instead of $\frac{3}{4} + \frac{1}{6}$, write it as:

$\frac{\textit{something}}{12} + \frac{\textit{something}}{12}$. Then you will see what the correct answer is.

To find out the "somethings", start with the first fraction. If you are changing the denominator in the fraction $\frac{3}{4}$ to $\frac{"\textit{something}"}{12}$, you multiplied 4 by 3 to get 12. So multiply the numerator by 3 also. That gives us $\frac{9}{12}$.

Pupil : OK, now I'll try the second fraction, $\frac{1}{6}$. If I change the denominator to 12, I multiplied 6 by 2 to get 12. So I will multiply the numerator (1) by 2, which is 2. So that gives me a fraction of $\frac{2}{12}$.

Teacher : Right so far. We just made the problem easier to solve by converting it this way:

$\frac{3}{4} + \frac{1}{6}$

Become

$\frac{9}{12} + \frac{2}{12}$

Pupil : That's much easier to solve. 9 + 2 = 11, so the answer would be $\frac{11}{12}$.

Teacher : Yes! It works the same way for subtraction. Try this problem: $\frac{4}{5} - \frac{4}{15}$.

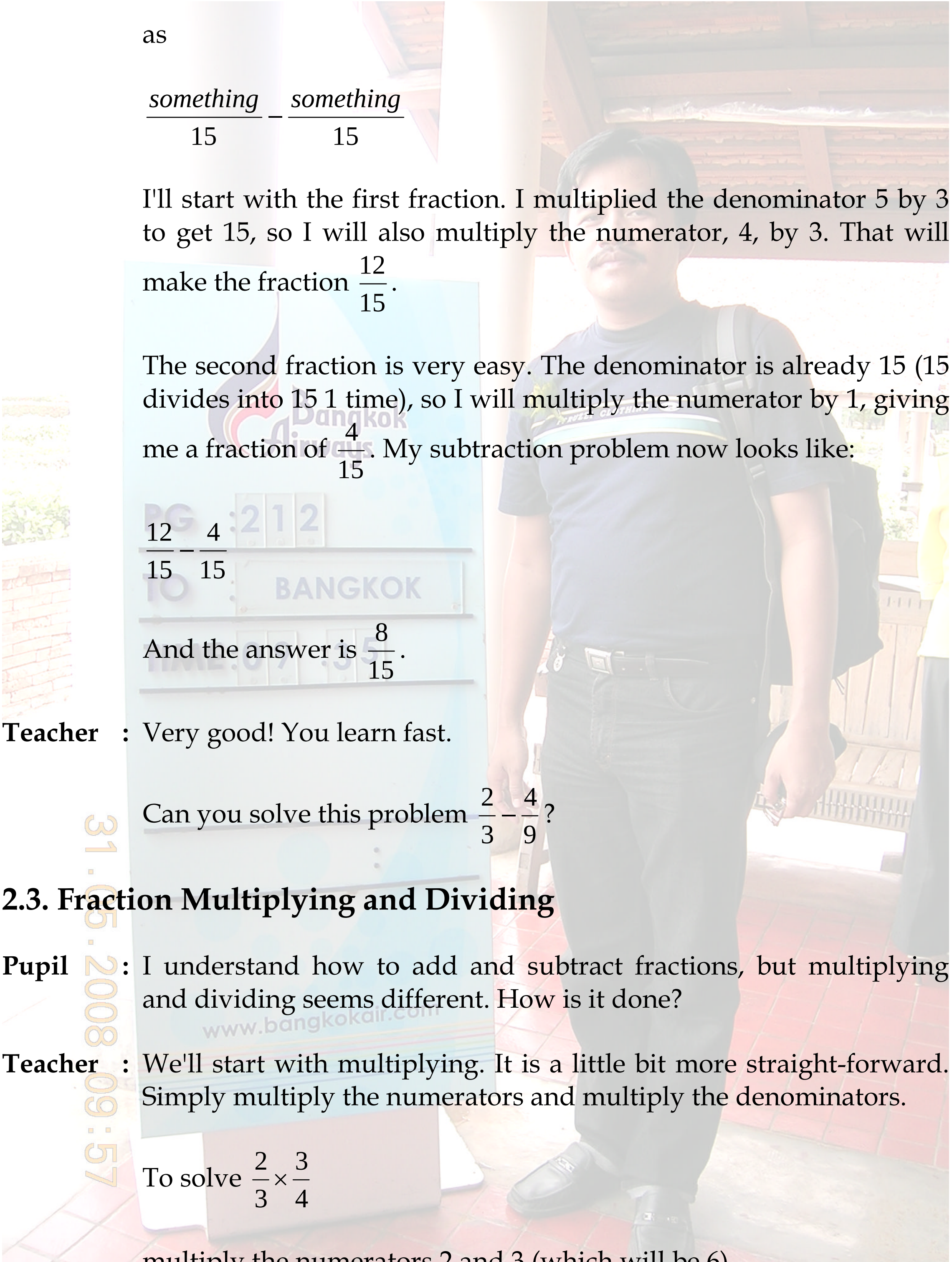
Pupil : Let's see. The lowest common denominator is 15, because both 5 and 15 divide evenly into it.

So I can rewrite the problem:

$$\frac{4}{5}-\frac{4}{15}$$

as

$$\frac{something}{15}-\frac{something}{15}$$



I'll start with the first fraction. I multiplied the denominator 5 by 3 to get 15, so I will also multiply the numerator, 4, by 3. That will make the fraction $\frac{12}{15}$.

The second fraction is very easy. The denominator is already 15 (15 divides into 15 1 time), so I will multiply the numerator by 1, giving me a fraction of $\frac{4}{15}$. My subtraction problem now looks like:

$$\frac{12}{15}-\frac{4}{15}$$

And the answer is $\frac{8}{15}$.

Teacher : Very good! You learn fast.

Can you solve this problem $\frac{2}{3}-\frac{4}{9}$?

2.3. Fraction Multiplying and Dividing

Pupil : I understand how to add and subtract fractions, but multiplying and dividing seems different. How is it done?

Teacher : We'll start with multiplying. It is a little bit more straight-forward. Simply multiply the numerators and multiply the denominators.

To solve $\frac{2}{3}\times\frac{3}{4}$

multiply the numerators 2 and 3 (which will be 6)

and then the denominators 3 and 4 (which will be 12) so, $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$

Pupil : And I already learned that I can reduce that fraction to $\frac{1}{2}$ by dividing both numbers by 6. How does it work when I divide numbers?

Teacher : Let's try it with the problem $\frac{2}{7}$ divided by $\frac{2}{5}$. There are several ways to look at this, but one way is to invert the fraction doing the dividing, and then multiply the numbers across. Invert means to turn upside down, so the steps of the problem look like this:

Step 1 : $\frac{2}{7}$ divided by $\frac{2}{5}$

Step 2 : inverting $\frac{2}{5}$ makes it $\frac{2}{7} \times \frac{5}{2}$

Step 3 : multiplying makes it $\frac{10}{14}$.

Step 4 : reducing makes it $\frac{5}{7}$

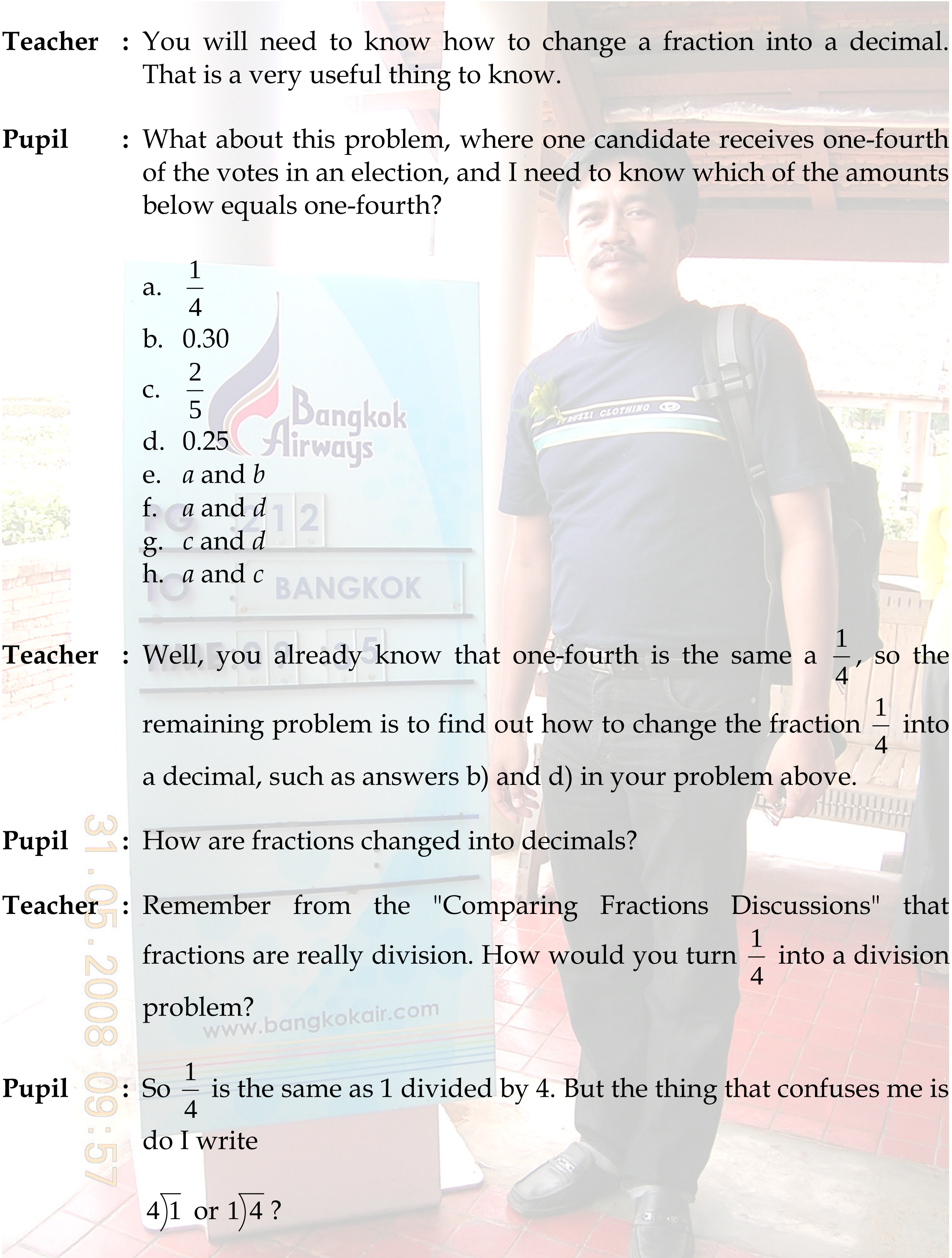
Pupil : OK, so $\frac{2}{7}$ divided by $\frac{2}{5}$ equals $\frac{5}{7}$.

Teacher : That's right.

Now try this one: $\frac{1}{3}$ divided by $\frac{6}{7}$.

3. Decimals and Percents Discussion

Pupil : What if I am solving a problem where some of the numbers are decimals instead of fractions?



Teacher : You will need to know how to change a fraction into a decimal. That is a very useful thing to know.

Pupil : What about this problem, where one candidate receives one-fourth of the votes in an election, and I need to know which of the amounts below equals one-fourth?

a. $\frac{1}{4}$
b. 0.30
c. $\frac{2}{5}$
d. 0.25
e. *a* and *b*
f. *a* and *d*
g. *c* and *d*
h. *a* and *c*

Teacher : Well, you already know that one-fourth is the same a $\frac{1}{4}$, so the remaining problem is to find out how to change the fraction $\frac{1}{4}$ into a decimal, such as answers b) and d) in your problem above.

Pupil : How are fractions changed into decimals?

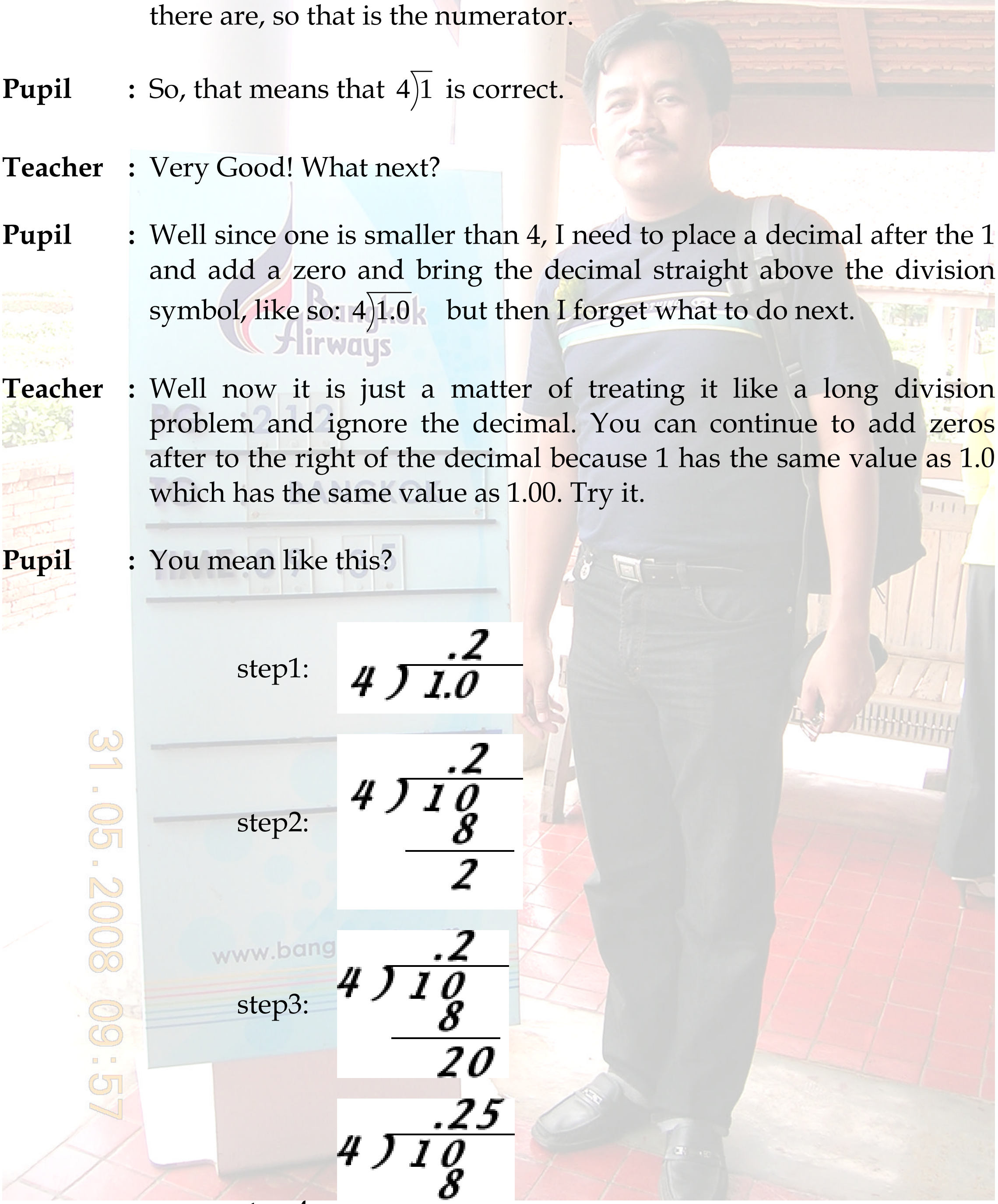
Teacher : Remember from the "Comparing Fractions Discussions" that fractions are really division. How would you turn $\frac{1}{4}$ into a division problem?

Pupil : So $\frac{1}{4}$ is the same as 1 divided by 4. But the thing that confuses me is do I write

$4\overline{)1}$ or $1\overline{)4}$?

Teacher : Let's review what the numbers mean. In a fraction, the denominator represents how many equal sized pieces a group or a whole is broken into; which in this problem is 4. The numerator then represents how many of those equal sized pieces there are, in this case 1. If you think about it this way, you can see the similarity between the idea of a fraction and concept of division.

Now let's see how the fraction notation can be represented as division. The number on the outside of the division symbol represents how many equal sized pieces we are trying to divide the whole into, i.e the denominator. The number on the inside of the division symbol represents how many of these equal sized pieces there are, so that is the numerator.



Pupil : So, that means that $4\overline{)1}$ is correct.

Teacher : Very Good! What next?

Pupil : Well since one is smaller than 4, I need to place a decimal after the 1 and add a zero and bring the decimal straight above the division symbol, like so: $4\overline{)1.0}$ but then I forget what to do next.

Teacher : Well now it is just a matter of treating it like a long division problem and ignore the decimal. You can continue to add zeros after to the right of the decimal because 1 has the same value as 1.0 which has the same value as 1.00. Try it.

Pupil : You mean like this?

step1:

$$\begin{array}{r} .2 \\ 4\overline{)1.0} \end{array}$$

step2:

$$\begin{array}{r} .2 \\ 4\overline{)10} \\ 8 \\ \hline 2 \end{array}$$

step3:

$$\begin{array}{r} .2 \\ 4\overline{)10} \\ 8 \\ \hline 20 \end{array}$$

step 4:

$$\begin{array}{r} .25 \\ 4\overline{)10} \\ 8 \\ \hline 20 \\ 20 \\ \hline 0 \end{array}$$

Teacher : You've got it. So now that you know how to convert a fraction to a decimal, can you figure out the answer to our original question?

Pupil : Sure. Now that I know $\frac{1}{4}$ = 0.25, if I do the same process to $\frac{2}{5}$ I get $\frac{2}{5}$ = 0.4 Since the decimal values are different, the fractions are different too!

Pupil : How are fractions and percents related to each other?

Teacher : Both fractions and percents represent parts of a whole. Where a fraction is a part of the amount of the denominator, a percent is always some part of 100. The word "percent" comes from Latin and means, "part of one hundred".

Pupil : So if I have fifty parts of one hundred, that is fifty percent, or 50%, right?

Teacher : Yes, and to express 50% as a fraction, picture it this way:

$\frac{50}{100}$ = Fifty parts of one hundred

Pupil : OK, that makes sense. And it looks like I can reduce that fraction by dividing both numbers by 50.

$\frac{50}{100}$

50 divided by 50 = 1

100 divided by 50 = 2

So, the reduced fraction is $\frac{1}{2}$

Teacher : Good work! Fifty percent is one half. Now let's try another. Suppose I have a fraction of $\frac{1}{5}$. How would I convert that to a percent?

Pupil : Well, if a percent is some part of 100, then I will multiply $\frac{1}{5}$ by 100.

$\frac{1}{5}\times 100$ is the same as $\frac{1}{5}\times\frac{100}{1}$

And I multiply across:

1 x 100 = 100 in the numerator
5 x 1 = 5 in the denominator
and $\frac{100}{5} = 20$

Teacher : Yes. So $\frac{1}{5}$ is the same as 20%.